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Primordial Density Perturbations and Reheating from Gravity

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ABSTRACT

We consider the presence and evolution of primordial density perturbations in a cosmological model based on a simple *ansatz* which captures – by providing a set of effective gravitational field equations – the strength of the enhanced quantum loop effects that can arise during inflation. After deriving the general equations that perturbations obey, we concentrate on scalar perturbations and show that their evolution is quite different than that of conventional inflationary models but still phenomenologically acceptable. The main reason for this novel evolution is the presence of an oscillating regime after the end of inflation which makes *all* super-horizon scalar modes oscillate. The same reason allows for a natural and very fast reheating mechanism for the universe.

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1 Introduction

During the inflationary era infrared gravitons are produced out of the vacuum because of the accelerated expansion of spacetime. Such a production can only occur for particles that are light compared to the Hubble scale without classical conformal invariance; gravitons and massless minimally coupled scalars are unique in that respect.

The self-gravitation of the vast ensemble of inflationary gravitons must act to slow the expansion rate [1]. This effect is inherently non-local because it couples the local graviton energy density with the potential induced by the interaction stress of the gravitons throughout the past light-cone. This suggests that the relevant effective field equations should be non-local. Non-local models of cosmology have been much studied because they can avoid the problem that de Sitter must be a solution for any local, stable theory, and because non-local couplings between different times can ease fine tuning problems [2, 3].

Quantum gravitational loop corrections which can be computed during inflation grow like the logarithm of the inflationary scale factor [4, 5, 6, 7]. It should ultimately be possible to derive the most cosmologically significant part of the effective field equations by summing the series of leading infrared logarithms. Starobinsky has proposed a technique for accomplishing this [8], and Starobinsky and Yokoyama have applied it to scalar potential models [9]. Starobinsky's method has recently been extended to Yukawa-coupled fermions [10] and to scalar quantum electrodynamics [11]. It has not yet been extended to quantum gravity but there are reasons for believing that some version of it can be [12].

Although full control of the effect requires a non-perturbative resummation technique, one can attempt to anticipate the results of such a formalism in a variety of ways. One approach is to simplify the full quantum gravitational dynamics by assuming that the exact graviton remains transverse-traceless and free, but propagates in the background geometry of an effective scale factor which is determined from the expectation value of the g_{00} gravitational constraint equation of motion [13]. The simplified theory retains the proper perturbative limit for de Sitter spacetime and may provide the basis for a tractable non-perturbative formulation.

Another approach is to use the physical principles responsible for the non-trivial quantum gravitational back-reaction on inflation and construct a phenomenological model which we can then directly evolve. In a previous pa-

per [14] we proposed a phenomenological model which can provide evolution beyond perturbation theory. In one sentence, we constructed an *effective* conserved stress-energy tensor $T_{\mu\nu}[g]$ which modifies the gravitational equations of motion: [1]

$$G_{\mu\nu} \;\equiv\; R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \;=\; -\Lambda\, g_{\mu\nu} \,+\, 8\pi G\, T_{\mu\nu}[g] \;\; . \tag{1}$$

and which, we hope, contains the most cosmologically significant part of the full effective quantum gravitational equations.

Our physical *ansatz* consists of parametrizing $T_{\mu\nu}[g]$ as a "perfect fluid":

$$T_{\mu\nu}[g] \;=\; (\rho + p)\, u_\mu\, u_\nu \,+\, p\, g_{\mu\nu} \;\; , \tag{2}$$

with the gravitationally induced pressure given as the following functional of the metric tensor:

$$p[g](x) \;=\; \Lambda^2\, f[-\epsilon\, X](x) \qquad , \qquad X \;\equiv\; \frac{1}{\square} R \;\; , \tag{3}$$

where the function f satisfies:

$$f[-\epsilon\, X] \;=\; -\epsilon\, X \,+\, O(\epsilon^2) \;\; , \tag{4}$$

and where the scalar d'Alembertian:

$$\square \;\equiv\; \frac{1}{\sqrt{-g}}\, \partial_\mu \Big(g^{\mu\nu} \sqrt{-g}\, \partial_\nu \Big) \;\; , \tag{5}$$

is defined with retarded boundary conditions. The induced energy density $\rho[g]$ and 4-velocity $u_\mu[g]$ are determined, up to their initial value data, from stress-energy conservation:

$$D^\mu\, T_{\mu\nu} \;=\; 0 \;\; , \tag{6}$$

which implies:

$$u \cdot \partial\, p \;=\; D_\mu \Big[\, (\rho + p) u^\mu \, \Big] \;\; , \tag{7}$$

$$u \cdot \partial\, \rho \;=\; -(D \cdot u)(\rho + p) \;\; , \tag{8}$$

$$(\rho + p)\; u \cdot D u_\nu \;=\; -(\, \partial_\nu \,+\, u_\nu\, u \cdot \partial \,)\, p \;\; . \tag{9}$$

---

[1] Hellenic indices take on spacetime values while Latin indices take on space values. Our metric tensor $g_{\mu\nu}$ has spacelike signature and our curvature tensor equals: $R^\alpha{}_{\beta\mu\nu} \equiv \Gamma^\alpha{}_{\nu\beta,\mu} + \Gamma^\alpha{}_{\mu\rho}\, \Gamma^\rho{}_{\nu\beta} - (\mu \leftrightarrow \nu)$. The initial value of the Hubble parameter is $3 H_0^2 \equiv \Lambda$. We restrict our analysis to scales $M \equiv (\, \Lambda / 8\pi G\,)^{\frac{1}{4}}$ below the Planck mass $M_{\rm Pl} \equiv G^{-1}$ so that the dimensionless coupling constant $\epsilon \equiv G\Lambda$ of the theory is small.

The 4-velocity is chosen to be timelike and normalized:

$$g^{\mu\nu} \, u_\mu u_\nu = -1 \qquad \Longrightarrow \qquad u^\mu \, u_{\mu;\nu} = 0 \ . \tag{10}$$

The purpose of this paper is to study the behaviour of the primordial scalar perturbations in the phenomenological model summarized above. Due to the non-local structure of the model the standard perturbation analysis must be extended and this is done in Section 3 after the relevant cosmological background is presented in Section 2. The evolution and novel features of the scalar perturbations are addressed in Section 4 and their normalization in Section 5. A result of the novel behaviour of our scalar mode functions after the end of the inflationary era allows a natural and very fast mechanism for reheating the universe which is described in Section 6. Our conclusions comprise Section 7.

2 The Cosmological Background

The large-scale homogeneity and isotropy of the universe selects Friedman-Robertson-Walker (FRW) spacetimes as those of primary cosmological interest; their line element for zero spatial curvature equals in co-moving coordinates:

$$d\bar{s}^2 \;=\; \bar{g}_{\mu\nu}(t) \, dx^\mu dx^\nu \;=\; -dt^2 \,+\, a^2(t) \, d\mathbf{x} \cdot d\mathbf{x} \ . \tag{11}$$

Derivatives of the scale factor $a(t)$ give the Hubble parameter $H(t)$ – a measure of the cosmic expansion rate – and the deceleration parameter $q(t)$ – a measure of the cosmic acceleration:

$$H(t) \;\equiv\; \frac{\dot{a}(t)}{a(t)} = \frac{d}{dt} \ln a(t) \ , \tag{12}$$

$$q(t) \;\equiv\; -\frac{a(t)\,\ddot{a}(t)}{\dot{a}^2(t)} = -1 \,-\, \frac{\dot{H}(t)}{H^2(t)} \ . \tag{13}$$

For these spacetimes the stress-energy tensor (2) equals:

$$\bar{T}_{00} \;=\; \bar{u}_0 \, \bar{u}_0 \, (\bar{\rho} + \bar{p}) \,-\, \bar{p} \;=\; \bar{\rho} \ , \tag{14}$$

$$\bar{T}_{0i} \;=\; 0 \ , \tag{15}$$

$$\bar{T}_{ij} \;=\; \bar{u}_i \, \bar{u}_j \, (\bar{\rho} + \bar{p}) \,+\, \bar{g}_{ij} \, \bar{p} \;=\; \bar{g}_{ij} \, \bar{p} \ . \tag{16}$$

An immediate consequence of isotropy and the normalization condition (10) is:

$$\bar{u}_\mu \;=\; -\,\delta_\mu^{\;\;0} \qquad \Longleftrightarrow \qquad \bar{u}^\mu \;=\; \delta^\mu_{\;\;0} \;\; . \tag{17}$$

The Ricci tensor and Ricci scalar become, respectively:

$$\bar{R}_{00} \;=\; -\left[\frac{3\ddot{a}}{a}\right] \;=\; -\Big(3H^2 \,+\, 3\dot{H}\Big) \;\;, \tag{18}$$

$$\bar{R}_{0i} \;=\; 0 \;\;, \tag{19}$$

$$\bar{R}_{ij} \;=\; \left[\frac{\ddot{a}}{a} \,+\, \frac{2\dot{a}^2}{a^2}\right]\bar{g}_{ij} \;=\; \Big(3H^2 \,+\, \dot{H}\Big)\,\bar{g}_{ij} \;\;, \tag{20}$$

and:

$$\bar{R} \;=\; \left[\frac{6\ddot{a}}{a} \,+\, \frac{6\dot{a}^2}{a^2}\right] \;=\; \Big(12H^2 \,+\, 6\dot{H}\Big) \;\;. \tag{21}$$

In view of (14-16, 18-21), the gravitational equations of motion (1) take the form:

$$3H^2 \;=\; \Lambda \,+\, 8\pi G\,\bar{\rho} \;\;, \tag{22}$$

$$-2\dot{H} - 3H^2 \;=\; -\Lambda \,+\, 8\pi G\,\bar{p} \;\;, \tag{23}$$

while the conservation equation (6) becomes:

$$\dot{\bar{\rho}} \;=\; -3H\,(\bar{\rho} + \bar{p}) \;\;. \tag{24}$$

The latter implies that:

$$\bar{\rho}(t) \;=\; -\bar{p}(t) \,+\, \frac{1}{a^3(t)}\int_0^t dt'\; a^3(t')\,\dot{\bar{p}}(t') \;\;. \tag{25}$$

When acting on functions which only depend on co-moving time, the scalar d'Alembertian (5) for FRW geometries equals:

$$\bar{\square} \;=\; -\Big(\partial_t^2 \,+\, 3H\partial_t\Big) \;\;, \tag{26}$$

so that its inverse is:

$$\frac{1}{\bar{\square}} \;=\; -\int_0^t dt'\,\frac{1}{a^3(t')}\int_0^{t'} dt''\; a^3(t'') \;\;. \tag{27}$$

Consequently, the source $\bar{X}$ can be written as follows:

$$\bar{X} \;=\; \frac{1}{\bar{\square}}\,\bar{R} \;=\; -\int_0^t dt'\,\frac{1}{a^3(t')}\int_0^{t'} dt''\; a^3(t'')\,\Big[12H^2(t'') \,+\, 6\dot{H}^2(t'')\Big] \;\;. \tag{28}$$

Note that we have taken the initial time to be at $t = 0$.

3 Perturbations

Small deviations from the homogeneity and isotropy of the FRW geometries are necessary to address, among other issues, the existence of primordial density perturbations. To account for these deviations, we first define small but general perturbations of all relevant variables about their FRW values. In co-moving coordinates we have:

$$
\begin{aligned}
p(t,\mathbf{x}) &\equiv \bar{p}(t) + \Delta p(t,\mathbf{x}) \ , & (29)\\
\rho(t,\mathbf{x}) &\equiv \bar{\rho}(t) + \Delta\rho(t,\mathbf{x}) \ , & (30)\\
u_0(t,\mathbf{x}) &\equiv \bar{u}_0 + \Delta u_0(t,\mathbf{x}) \ , & (31)\\
u_i(t,\mathbf{x}) &\equiv \bar{u}_i + a(t)\,\Delta u_i(t,\mathbf{x}) \ , & (32)\\
g_{00}(t,\mathbf{x}) &\equiv \bar{g}_{00} + h_{00}(t,\mathbf{x}) = -1 + h_{00}(t,\mathbf{x}) \ , & (33)\\
g_{0i}(t,\mathbf{x}) &\equiv \bar{g}_{0i} + a(t)\,h_{0i}(t,\mathbf{x}) = a(t)\,h_{0i}(t,\mathbf{x}) \ , & (34)\\
g_{ij}(t,\mathbf{x}) &\equiv \bar{g}_{ij}(t) + a^2(t)\,h_{ij}(t,\mathbf{x}) = a^2(t)\,[\,\delta_{ij} + h_{ij}(t,\mathbf{x})\,] \ . & (35)
\end{aligned}
$$

We then substitute (29-35) in the gravitational equations of motion (1):

$$
\begin{aligned}
G^{\mu}{}_{\nu} &\equiv R^{\mu}{}_{\nu} - \frac{1}{2}\,\delta^{\mu}{}_{\nu}\,R \\
&= -\Lambda\,\delta^{\mu}{}_{\nu} + 8\pi G\left[\,(\rho+p)\,u^{\mu}\,u_{\nu} + p\,\delta^{\mu}{}_{\nu}\right] \equiv -\Lambda\,\delta^{\mu}{}_{\nu} + T^{\mu}{}_{\nu} \ , \qquad (36)
\end{aligned}
$$

and obtain equations to the desired order in the perturbation; for our purposes, the first order equations will suffice. This is a tedious process which to some degree can be simplified by a proper choice of coordinate system and field variables. It turns out that conformal coordinates $(\eta,\mathbf{x})$ and a set of gauge invariant variables is the optimal choice [16].

In conformal coordinates – which we shall use thereafter in this Section – the background invariant element is proportional to that of flat spacetime:

$$
d\bar{s}^2 = \bar{g}_{\mu\nu}(\eta)\,dx^{\mu}dx^{\nu} = a^2(\eta)\left(-\,d\eta^2 + d\mathbf{x}\cdot d\mathbf{x}\right) \ . \qquad (37)
$$

The relation between co-moving and conformal times is:

$$
d\eta = \frac{dt}{a(t)} \qquad \Longrightarrow \qquad \frac{d}{d\eta} = a\,\frac{d}{dt} \ . \qquad (38)
$$

The corresponding perturbations take the form:

$$
p(\eta,\mathbf{x}) \equiv \bar{p}(\eta) + \Delta p(\eta,\mathbf{x}) \ , \qquad (39)
$$

$$
\begin{aligned}
\rho(\eta,\mathbf{x}) \;&\equiv\; \bar{\rho}(\eta) \,+\, \Delta\rho(\eta,\mathbf{x}) \;\;, && (40)\\
u_\mu(\eta,\mathbf{x}) \;&\equiv\; \bar{u}_\mu(\eta) \,+\, a(\eta)\,\Delta u_\mu(\eta,\mathbf{x}) \;=\; a(\eta)\,[\,-\delta_\mu^{\;0} \,+\, \Delta u_\mu(\eta,\mathbf{x})\,] \;\;, && (41)\\
g_{\mu\nu}(\eta,\mathbf{x}) \;&\equiv\; \bar{g}_{\mu\nu}(\eta) \,+\, a^2(\eta)\,h_{\mu\nu}(t,\mathbf{x}) \;=\; a^2(\eta)\,[\,\eta_{\mu\nu} + h_{\mu\nu}(t,\mathbf{x})\,] \;\;, && (42)
\end{aligned}
$$

where we have used the conformal analogue of (17):

$$
\bar{u}_\mu \;=\; -a\,\delta_\mu^{\;0} \qquad \Longleftrightarrow \qquad \bar{u}^\mu \;=\; a^{-1}\,\delta^\mu_{\;0} \;\;. \tag{43}
$$

- **The Left Hand Side**

As a result of straightforward manipulations, the left hand side of the field equations (36) can be written as:

$$
G^\mu_{\;\nu} \;\equiv\; \bar{G}^\mu_{\;\nu} + \Delta G^\mu_{\;\nu} \;\;, \tag{44}
$$

with: [2]

$$
\bar{G}^\mu_{\;\nu} \;=\; \delta^\mu_{\;0}\,\delta^\nu_{\;0}\left[2\,\frac{a''}{a^3} - 4\,\frac{a'^{\,2}}{a^4}\right] \,+\, \delta^\mu_{\;\nu}\left[-2\,\frac{a''}{a^3} + \frac{a'^{\,2}}{a^4}\right] \;\;, \tag{45}
$$

$$
\begin{aligned}
\Delta G^\mu_{\;\nu} \;=\; &-\delta^0_{\;\nu}\,h^{\mu\nu}\left[-2\,\frac{a''}{a^3} + \frac{a'^{\,2}}{a^4}\right] \,+\, \delta^\mu_{\;\nu}\,h^{00}\left[-2\,\frac{a''}{a^3} + \frac{a'^{\,2}}{a^4}\right]\\
&-(\,\eta^{\mu\rho}\,\delta^\sigma_{\;\nu} - \delta^\mu_{\;\nu}\,\eta^{\rho\sigma}\,)\left[\,h_{0\rho,\sigma} + h_{0\sigma,\rho} - h'_{\rho\sigma}\,\right]\frac{a'}{a^3}\\
&+\left[\,h^{\mu\rho}_{\;\;\;,\rho\nu} + h_{\nu\rho}^{\;\;\;,\rho\mu} - h^\mu_{\;\nu}{}^{,\rho}{}_{\rho} - h^\rho_{\;\rho}{}^{,\mu}{}_{\nu}\,\right]\frac{1}{2a^2}\\
&-\delta^\mu_{\;\nu}\left[\,h^{\rho\sigma}_{\;\;\;,\rho\sigma} - h^\rho_{\;\rho}{}^{,\sigma}{}_{\sigma}\,\right]\frac{1}{2a^2} \;\;.
\end{aligned} \tag{46}
$$

It is convenient to 3+1 decompose the background value $\bar{G}^\mu_{\;\nu}$ in (45):

$$
\begin{aligned}
\bar{G}^0_{\;0} \;&=\; -3\,\frac{a'^{\,2}}{a^4} \;\;, && (47)\\
\bar{G}^0_{\;i} \;&=\; 0 \;\;, && (48)\\
\bar{G}^i_{\;j} \;&=\; \left[-2\,\frac{a''}{a^3} + \frac{a'^{\,2}}{a^4}\right]\delta_{ij} \;\;, && (49)
\end{aligned}
$$

[2] In conformal coordinates, the prime superscript denotes differentiation with respect to conformal time.

as well as the first order perturbation $\Delta G^{\mu}_{\ \nu}$ in (46):

$$\Delta G^{0}_{\ 0} \ = \ -3\,\frac{a'^{\,2}}{a^4}\,h_{00} \ + \ \frac{a'}{a^3}\left[2h_{0i,i} - h'_{ii}\right] + \frac{1}{2a^2}\left[\nabla^2 h_{ii} - h_{ij,ij}\right] \ \ , \tag{50}$$

$$\Delta G^{0}_{\ i} \ = \ \frac{a'}{a^3}\,h_{00,i} \ + \ \frac{1}{2a^2}\left[-h_{0j,ij} + \nabla^2 h_{0i} - h'_{ij,ij} + h'_{jj,i}\right] \ \ , \tag{51}$$

$$\begin{aligned}\Delta G^{i}_{\ j} \ = \ \delta_{ij}\Bigg\{ & \left[-2\,\frac{a''}{a^3} + \frac{a'^{\,2}}{a^4}\right] h_{00} \ + \ \frac{a'}{a^3}\left[-\,h'_{00} + 2h_{0k,k} - h'_{kk}\right] \\ & + \frac{1}{2a^2}\left[-\,h''_{kk} + \nabla^2 h_{kk} + 2h'_{0k,k} - h_{k\ell,k\ell} - \nabla^2 h_{00}\right]\Bigg\} \\ & + \frac{a'}{a^3}\left[-\,2h_{0(i,j)} + h'_{ij}\right] \\ & + \frac{1}{a^2}\left[h''_{ij} - \nabla^2 h_{ij} - 2h'_{0(i,j)} + 2h_{k(i,j)k} + h_{00,ij} - h_{kk,ij}\right] \ .\end{aligned} \tag{52}$$

⋆ *Identities*

In addition to the equations presented throughout the main text, various of the following expressions have been used to obtain the results of this subsection:

$$a_{,\rho} \ = \ a'\,\delta^{0}_{\ \rho} \qquad , \qquad a_{,\rho\sigma} \ = \ a''\,\delta^{0}_{\ \rho}\,\delta^{0}_{\ \sigma} \ \ , \tag{53}$$

$$\Gamma^{\rho}_{\ \mu\nu} \ \equiv \ \bar{\Gamma}^{\rho}_{\ \mu\nu} \ + \ \Delta\Gamma^{\rho}_{\ \mu\nu} \ \ , \tag{54}$$

$$\bar{\Gamma}^{\rho}_{\ \mu\nu} \ = \ \frac{a'}{a}\left[\delta^{\rho}_{\ \mu}\,\delta^{0}_{\ \nu} + \delta^{\rho}_{\ \nu}\,\delta^{0}_{\ \mu} - \eta^{\rho 0}\,\eta_{\mu\nu}\right] \ \ , \tag{55}$$

$$\Delta\Gamma^{\rho}_{\ \mu\nu} \ = \ \frac{a'}{a}\left[\eta_{\mu\nu}\,h^{\rho 0} - \eta^{\rho 0}\,h_{\mu\nu}\right] \ + \ \frac{1}{2}\left[h^{\rho}_{\ \mu,\nu} + h^{\rho}_{\ \nu,\mu} - h_{\mu\nu}^{\ \ \ ,\rho}\right] \ \ , \tag{56}$$

- **The Right Hand Side**

We write the gravitationally induced stress-energy tensor $T^{\mu}_{\ \nu}$ as:

$$T^{\mu}_{\ \nu} \ \equiv \ \bar{T}^{\mu}_{\ \nu} \ + \ \Delta T^{\mu}_{\ \nu} \ \ . \tag{57}$$

Its background value $\bar{T}^{\mu}_{\ \nu}$ is:

$$\bar{T}^{\mu}_{\ \nu} \ = \ (\bar{\rho} + \bar{p})\,\bar{u}^{\mu}\,\bar{u}_{\nu} \ + \ \bar{p}\,\delta^{\mu}_{\ \nu} \ \ , \tag{58}$$

and, using (43), its 3+1 decomposition takes the form:

$$\bar{T}^{0}_{\ 0} \ = \ -\bar{\rho} \qquad , \qquad \bar{T}^{0}_{\ i} \ = \ 0 \qquad , \qquad \bar{T}^{i}_{\ j} \ = \ \bar{p}\,\delta_{ij} \ \ . \tag{59}$$

The perturbation $\Delta T^{\mu}{}_{\nu}$ defined in (57) equals:

$$\Delta T^{\mu}{}_{\nu} \; = \; \bar{u}^{\mu} \, \bar{u}_{\nu} \, \Delta(\rho + p) \, + \, (\bar{\rho} + \bar{p}) \, \Delta(u^{\mu} \, u_{\nu}) \, + \, \Delta p \, \delta^{\mu}{}_{\nu} \;\; , \tag{60}$$

and, in view of (43), the 3+1 decomposition is given by:

$$\Delta T^{0}{}_{0} \; = \; -\Delta\rho \, + \, (\bar{\rho} + \bar{p}) \Big[- a \, \Delta u^{0} + a^{-1} \, \Delta u_{0} \Big] \;\; , \tag{61}$$

$$\Delta T^{0}{}_{i} \; = \; (\bar{\rho} + \bar{p}) \, a^{-1} \, \Delta u_{i} \;\; , \tag{62}$$

$$\Delta T^{i}{}_{j} \; = \; \Delta p \, \delta^{i}{}_{j} \;\; . \tag{63}$$

* *The induced pressure deviation* Δp

The starting point for the explicit computations is the induced pressure *ansatz* (3) which we shall expand to first order about the background geometry and determine Δp. Knowledge of Δp and use of the conservation equations will allow us to obtain the remaining deviations $\Delta\rho$ and Δu_{μ}, up to initial value data. [3] The aforementioned expansion is:

$$p \; = \; \Lambda^{2} \, f[-\epsilon \, X] \; = \; \Lambda^{2} \Big\{ f[-\epsilon \, \bar{X}] \, - \, f'[-\epsilon \, \bar{X}] \, (\epsilon \, \Delta X) \, + \, O(\epsilon^{2}) \Big\} \;\; , \tag{64}$$

where:

$$f'[-\epsilon \, \bar{X}] \; \equiv \; -\frac{1}{\epsilon} \, \frac{d}{d\bar{X}} f[-\epsilon \, \bar{X}] \;\; . \tag{65}$$

Therefore:

$$\bar{p} \; = \; \Lambda^{2} \, f[-\epsilon \, \bar{X}] \qquad , \qquad \bar{X} \; = \; \bar{\Box}^{-1} \bar{R} \;\; , \tag{66}$$

$$\Delta p \; = \; -\epsilon \Lambda^{2} \, f'[-\epsilon \, \bar{X}] \, \Delta X \;\; . \tag{67}$$

Now the first order perturbation ΔX equals:

$$\Delta X \; \equiv \; \Delta(\Box^{-1} R) \; = \; \frac{1}{\bar{\Box}} \Big[\, \Delta R - (\Delta \Box) \, \bar{X} \, \Big] \;\; , \tag{68}$$

and we must evaluate:

(i) the Ricci scalar perturbation ΔR:

$$R \; \equiv \; \bar{R} \, + \, \Delta R \;\; , \tag{69}$$

[3] As we shall see later in this subsection, the perturbation Δu_0 can be computed independent of the conservation equations.

for which a straightforward calculation gives:

$$\bar{R} \;=\; 6\,\frac{a''}{a^3} \quad , \tag{70}$$

$$\begin{aligned}\Delta R \;=&\; 6\,\frac{a''}{a^3}\,h_{00} \;+\; 3\,\frac{a'}{a^3}\Big[\,h'_{00} - 2h_{0i,i} + h'_{ii}\,\Big] \\ &+ \frac{1}{a^2}\Big[\,\nabla^2 h_{00} - \nabla^2 h_{ii} - 2h'_{0i,i} + h''_{ii} + h_{ij,ij}\,\Big] \quad , \end{aligned} \tag{71}$$

(ii) the d'Alembertian perturbation $\Delta\Box$:

$$\Box \;\equiv\; \bar{\Box} + \Delta\Box \quad , \tag{72}$$

which directly follows from the definition (5):

$$\begin{aligned}\bar{\Box} \;=&\; \frac{1}{\sqrt{-\bar{g}}}\,\partial_\mu\Big(\bar{g}^{\mu\nu}\sqrt{-\bar{g}}\;\partial_\nu\Big) \;=\; \frac{1}{a^4}\,\partial_\mu\Big(a^2\,\eta^{\mu\nu}\,\partial_\nu\Big) \\ =&\; \frac{1}{a^2}\left[\nabla^2 - \partial_0^2 - 2\frac{a'}{a}\,\partial_0\right] \;=\; \frac{1}{a^2}\left[\nabla^2 - \frac{1}{a^2}\,\partial_0\,a^2\,\partial_0\right] \quad , \end{aligned} \tag{73}$$

$$\begin{aligned}\Delta\Box \;=&\; \Delta\Big(\frac{1}{\sqrt{-g}}\Big)\,\partial_\mu\Big[\bar{g}^{\mu\nu}\sqrt{-\bar{g}}\;\partial_\nu\Big] \;+\; \frac{1}{\sqrt{-\bar{g}}}\,\partial_\mu\Big[(\Delta g^{\mu\nu})\sqrt{-\bar{g}}\;\partial_\nu\Big] \\ &+ \frac{1}{\sqrt{-\bar{g}}}\,\partial_\mu\Big[\bar{g}^{\mu\nu}(\Delta\sqrt{-\bar{g}})\;\partial_\nu\Big] \end{aligned} \tag{74}$$

$$= \frac{1}{2a^2}\,\eta^{\mu\nu}\,(\partial_\mu h)\,\partial_\nu \;-\; \frac{1}{a^4}\,\partial_\mu\Big[a^2\,h^{\mu\nu}\,\partial_\nu\Big] \quad . \tag{75}$$

As a result, the action of $\Delta\Box$ on the background source $\bar{X}$ gives:

$$(\Delta\Box)\,\bar{X} \;=\; \frac{1}{a^2}\,\bar{X}'\Big[-\frac{1}{2}h' - h'_{00,0} + h_{0i,i}\Big] \;-\; h_{00}\,\frac{1}{a^4}\,\partial_0\,a^2\,\partial_0\,\bar{X} \tag{76}$$

$$= \frac{1}{a^2}\,\bar{X}'\Big[-\frac{1}{2}h' - h'_{00,0} + h_{0i,i}\Big] \;+\; \bar{R}\,h_{00} \quad . \tag{77}$$

Subtracting equation (77) from equation (71) gives ΔX via (68) and, in turn, Δp via (67):

$$\begin{aligned}\Delta p \;=&\; -\,\epsilon\Lambda^2\,f'[-\epsilon\,\bar{X}] \times \frac{1}{\bar{\Box}}\Bigg\{\,3\,\frac{a'}{a^3}\Big[\,h'_{00} - 2h_{0i,i} + h'_{ii}\Big] \\ &+ \frac{1}{a^2}\Big[\,\nabla^2 h_{00} - \nabla^2 h_{ii} - 2h'_{0i,i} + h''_{ii} + h_{ij,ij}\Big] \\ &+ \frac{1}{a^2}\,\bar{X}'\Big[\,\frac{1}{2}h' + h'_{00,0} - h_{0i,i}\Big]\Bigg\} \quad . \end{aligned} \tag{78}$$

$*$ *The deviation* Δu_0

It is important to note that Δu_0 can be directly determined from the perturbation of the 4-velocity timelike condition (10):

$$g^{\mu\nu}\, u_\mu u_\nu \;=\; -1 \;\;\Longrightarrow\;\; (\Delta g^{\mu\nu})\, \bar{u}_\mu \bar{u}_\nu \;+\; 2g^{\mu\nu}\, \bar{u}_\mu\, (\Delta u_\nu) \;=\; 0 \tag{79}$$

$$\Longrightarrow\;\; -h_{00} \;+\; \frac{2}{a}\, \Delta u_0 \;=\; 0 \;\; . \tag{80}$$

We trivially conclude that:

$$\Delta u_0 \;=\; \frac{1}{2}\, a\, h_{00} \qquad , \qquad \Delta u^0 \;=\; \frac{1}{2a}\, h_{00} \;\; . \tag{81}$$

As a result, $\Delta T^0_{\;\;0}$ – given by equation (61) – is simplified:

$$\Delta T^0_{\;\;0} \;=\; -\Delta\rho \;\; . \tag{82}$$

$\star$ *Identities*

In addition to the equations presented throughout the main text, various of the following expressions have been used to obtain the results of this subsection:

$$\sqrt{-g} \;=\; \sqrt{-\bar{g}}\, \left[1 + \frac{1}{2}\, h + \ldots\right] \quad , \quad h \;\equiv\; h^\mu_{\;\;\mu} \;=\; -h_{00} + h_{ii} \;\; , \tag{83}$$

$$\bar{g}_{\mu\nu} \;=\; a^2\, \eta_{\mu\nu} \quad , \quad \bar{g}^{\mu\nu} \;=\; a^{-2}\, \eta^{\mu\nu} \;\; , \tag{84}$$

$$\sqrt{-\bar{g}} \;=\; a^4 \quad , \quad \frac{1}{\sqrt{-\bar{g}}} \;=\; \frac{1}{a^4} \;\; , \tag{85}$$

$$\Delta g_{\mu\nu} \;=\; a^2\, h_{\mu\nu} \quad , \quad \Delta g^{\mu\nu} \;=\; -a^{-2}\, h^{\mu\nu} \;\; , \tag{86}$$

$$\Delta\sqrt{-g} \;=\; \frac{1}{2}\, a^4\, h \quad , \quad \Delta\Big(\frac{1}{\sqrt{-g}}\Big) \;=\; -\frac{1}{2\, a^4}\, h \;\; . \tag{87}$$

$$\partial_\mu \bar{X} \;=\; \delta^0_{\;\;\mu}\, \bar{X}' \;\; . \tag{88}$$

When acting on functions that only depend on conformal time:

$$\bar{\Box} \;=\; -\frac{1}{a^4}\, \partial_0\, a^2\, \partial_0 \qquad , \qquad \frac{1}{\bar{\Box}} \;=\; -\frac{1}{\partial_0}\, \frac{1}{a^2}\, \frac{1}{\partial_0}\, a^4 \;\; . \tag{89}$$

- **The Conservation Equations**

Of the conservation equations (7-9), we shall use (7) and (9) to express $\Delta\rho$

and Δu_μ in terms of Δp. As a first step, we write them more explicitly:

$$\partial_\mu \Big[\sqrt{-g}\, (\rho + p)\, u^\mu \Big] \; = \; \sqrt{-g}\, u^\mu\, \partial_\mu\, p \;\; , \tag{90}$$

$$(\rho + p)\, u^\mu{}_{;\nu}\, u^\nu \; = \; -\Big[g^{\mu\nu} + u^\mu\, u^\nu \Big]\, \partial_\nu\, p \;\; , \tag{91}$$

and study them up to first order.

(i) Zeroth order

Because the background 4-velocity is given by (43), the background value of (90) equals:

$$\partial_0 \Big[a^3 (\, \bar\rho + \bar p \,) \Big] \; = \; a^3 \bar p' \;\; , \tag{92}$$

and integrates to:

$$\bar\rho + \bar p \; = \; \frac{1}{a^3} \int_{\eta_0}^{\eta} d\acute\eta\, a^3\, \bar p' \;\; , \tag{93}$$

which is the conformal analogue of equation (25). The background value of (91) leads to a tautology of the form $0 = 0$.

(ii) First order

When the perturbations defined in (39-42) are applied to the conservation equations (90-91) they respectively lead to:

$$\begin{aligned} &\partial_\mu \Big[(\Delta\sqrt{-g})\, (\bar\rho + \bar p)\, \bar u^\mu \; + \; \sqrt{-\bar g}\, \Big(\Delta(\rho + p) \Big)\, \bar u^\mu \; + \; \sqrt{-\bar g}\, (\bar\rho + \bar p)\, (\Delta u^\mu) \Big] \\ &\quad = \; (\Delta\sqrt{-g})\, \bar u^\mu\, \partial_\mu\, \bar p \; + \; \sqrt{-\bar g}\, (\Delta u^\mu)\, \partial_\mu\, \bar p \; + \; \sqrt{-\bar g}\, \bar u^\mu\, \partial_\mu (\Delta p) \;\; , \end{aligned} \tag{94}$$

$$\begin{aligned} &\Big(\Delta(\rho + p) \Big)\, \bar u^\mu{}_{;\nu}\, \bar u^\nu \; + \; (\bar\rho + \bar p)\, (\Delta u^\mu{}_{;\nu})\, \bar u^\nu \; + \; (\bar\rho + \bar p)\, \bar u^\mu{}_{;\nu}\, (\Delta u^\nu) \\ &\quad = \; -\Big[(\Delta g^{\mu\nu}) + \bar u^\mu\, (\Delta u^\nu) + (\Delta u^\mu)\, \bar u^\nu \Big]\, \partial_\nu\, \bar p \; - \; \Big[\bar g^{\mu\nu} + \bar u^\mu\, \bar u^\nu \Big]\, \partial_\nu (\Delta p) \;\; . \end{aligned} \tag{95}$$

Use of various background values and identities reduces equations (94-95) as follows:

$$\partial_0 \Big[a^3\, \Delta(\rho + p) \Big] \; = \; -\, a^4 (\bar\rho + \bar p) \Big[\partial_i\, \Delta u^i + \frac{1}{2a}\, h'_{ii} \Big] \; + \; a^3\, \Delta p' \;\; , \tag{96}$$

$$\begin{aligned} &a^3 (\bar\rho + \bar p)\, \Big\{ (\Delta u^\mu)' + 2\, \frac{a'}{a}\, \Delta u^\mu - \frac{a'}{a}\, \delta^\mu{}_0\, \Delta u^0 \; + \; \frac{1}{a}\, \eta^{\mu\nu} \Big[h_{0\nu,0} - \frac{1}{2} h_{00,\nu} \Big] \Big\} \\ &\quad = \; a^2\, h^{0\mu}\, \bar p' \; - \Big[\Delta u^\mu + \delta^\mu{}_0\, \Delta u^0 \Big] a^3\, \bar p' \; - \; a^2\, (\eta^{\mu\nu} - \delta^\mu{}_0\, \delta^\nu{}_0)\, \partial_\nu (\Delta p) \;\; . \end{aligned} \tag{97}$$

The last equation can be 3+1 decomposed and further reduced:

$$\Big\{\Big[\,a^3(\bar\rho+\bar p)\,\Big]^2(\Delta u^0)\Big\}' \;=\; \Big\{\Big[\,a^3(\bar\rho+\bar p)\,\Big]^2\frac{1}{2a}\,h_{00}\Big\}' \quad , \tag{98}$$

$$\begin{aligned}\Big[\,a^3(\bar\rho+\bar p)\,(a^2\,\Delta u^i)\,\Big]' \;&=\; -\Big[\,a^4(\bar\rho+\bar p)\,h_{0i}\,\Big]' \,+\, \frac{1}{2}\,a^4(\bar\rho+\bar p)\,\partial_i h_{00} \\ &\qquad -\,a^4\,\partial_i(\Delta p) \quad ,\end{aligned} \tag{99}$$

where we have extensively used (92). Equation (98) is trivially satisfied by the solution (81) for Δu^0. Given the perturbation Δp, equation (99) determines Δu^i up to its arbitrary initial value. Then, equation (96) has enough information at its disposal to determine $\Delta\rho$, again up to its arbitrary initial value. Therefore, $\Delta T^\mu_{\ \nu}$ has been completely specified to first order.

⋆ *Identities*

In addition to the equations presented throughout the main text, various of the following expressions have been used to obtain the results of this subsection:

$$\bar u^\mu_{\ ,\nu} \;=\; -\frac{a'}{a^2}\,\delta^\mu_{\ 0}\,\delta^0_{\ \nu} \quad , \tag{100}$$

$$\bar u^\mu_{\ ;\nu} \;=\; \bar u^\mu_{\ ,\nu} \,+\, \bar\Gamma^\mu_{\ \nu\rho}\,\bar u^\rho \;=\; \frac{a'}{a^2}\Big[\,\delta^\mu_{\ \nu} - \delta^\mu_{\ 0}\,\delta^0_{\ \nu}\,\Big] \quad , \quad \bar u^\mu_{\ ;\nu}\,\bar u^\nu \;=\; 0 \quad , \tag{101}$$

$$\Delta u_\mu \;=\; (\Delta g_{\mu\nu})\,\bar u^\nu + \bar g_{\mu\nu}\,(\Delta u^\nu) \;=\; a\,h_{\mu 0} + a^2\,\eta_{\mu\nu}\,(\Delta u^\nu) \quad , \tag{102}$$

$$\Delta u^\mu \;=\; (\Delta g^{\mu\nu})\,\bar u_\nu + \bar g^{\mu\nu}\,(\Delta u_\nu) \;=\; \frac{1}{a}\,h^{\mu 0} + \frac{1}{a^2}\,\eta^{\mu\nu}\,(\Delta u_\nu) \quad , \tag{103}$$

$$\Delta u^\mu_{\ ;\nu} \;=\; \Delta u^\mu_{\ ,\nu} + \bar\Gamma^\mu_{\ \nu\rho}\,(\Delta u^\rho) + (\Delta\Gamma^\mu_{\ \nu\rho})\,\bar u^\rho \quad , \tag{104}$$

$$(\Delta u^\mu_{\ ;\nu})\,\bar u^\nu \;=\; \frac{1}{a^2}\Big[a\,(\Delta u^\mu)\Big]' \,+\, \frac{1}{a^2}\,{h^\mu_{\ 0}}' \,-\, \frac{1}{2a^2}\,{h_{00}}^{,\mu} \quad , \tag{105}$$

$$\partial_\mu\,\bar p \;=\; \delta^0_{\ \mu}\,\bar p' \;=\; \delta^0_{\ \mu}\,\frac{1}{a^3}\Big[\,a^3(\rho+p)\,\Big]' \quad . \tag{106}$$

- **The Full Equations**

At the cost of being redundant, we list the results derived in this section for the gravitational equations of motion (36) up to first order in the perturbations (29-42).

(i) Background equations: We are given a background FRW spacetime characterized by the scale factor a, and an induced background pressure $\bar p$; then:

$$(00) \quad\Longrightarrow\quad -3\,\frac{a'^{\,2}}{a^4} \;=\; -\,3H_0^2 \,+\, 8\pi G\Big[\bar p - \frac{1}{a^3}\int_{\eta_0}^{\eta} d\acute\eta\; a^3\,\bar p'\,\Big] \quad , \tag{107}$$

$$(0i) \implies \quad 0 \;=\; 0 \;\;, \tag{108}$$

$$(ij) \implies \quad \left[-2\,\frac{a''}{a^3}+\frac{a'^{\,2}}{a^4}\right]\delta_{ij} \;=\; -\,3H_0^2\,\delta_{ij} \;+\; 8\pi G\,\bar{p}\,\delta_{ij} \;\;. \tag{109}$$

(ii) First order equations: We are given the spacetime perturbation $h_{\mu\nu}$ and the induced pressure perturbation Δp ; then: [4]

$$(00) \implies \quad \Delta G^0{}_0 \;=\; 8\pi G\,(-\Delta\rho) \;\;, \tag{110}$$

$$(0i) \implies \quad \Delta G^0{}_i \;=\; 8\pi G\;(\bar{\rho}+\bar{p})\,\frac{1}{a}\,\Delta u_i \;\;, \tag{111}$$

$$(ij) \implies \quad \Delta G^i{}_j \;=\; 8\pi G\;\Delta p\;\delta^i{}_j \;\;. \tag{112}$$

- **Scalar Perturbations**

The very nature of the gravitational equations of motion allows for scalar, vector and tensor perturbations. Of these, it is the scalar perturbations that have, up to now, the biggest phenomenological interest. A general scalar perturbation is characterized by four scalar functions $\phi,\ B,\ \psi,\ E$ which are defined as follows [16]:

$$h_{00} \;\equiv\; -2\phi \;\;, \tag{113}$$

$$h_{0i} \;\equiv\; -B_{,i} \;\;, \tag{114}$$

$$h_{ij} \;\equiv\; -2\psi\,\delta_{ij} \;-\; 2E_{,ij} \;\;, \tag{115}$$

and which can be combined into two gauge invariant scalar field variables [16]:

$$\Phi \;\equiv\; \phi \;-\; \frac{1}{a}\Big[\,a\,(B-E')\Big]' \;=\; \phi \;-\; \frac{a'}{a}\,(B-E') \;-\; (B'-E'') \;\;, \tag{116}$$

$$\Psi \;\equiv\; \psi \;+\; \frac{a'}{a}\,(B-E') \;\;. \tag{117}$$

To deduce the equations that are obeyed by the scalar perturbations we must isolate their contribution to the full gravitational equations using (113-115) and then express it in terms of the invariant variables (116-117):

[4]To avoid prohibitively long expressions we have not expressed all variables in terms of $h_{\mu\nu}$ and Δp.

(i) The left hand side

The 3+1 decomposition of $(\Delta G^{\mu}{}_{\nu})_S$ equals:

$$\Big(\Delta G^0{}_0\Big)_S \;=\; -\frac{2}{a^2}\,\nabla^2\Big[\psi+\frac{a'}{a}(B-E')\Big] \;+\; 6\,\frac{a'}{a^3}\Big[\psi'+\frac{a'}{a}\,\phi\Big] \tag{118}$$

$$\;=\; -\frac{2}{a^2}\,\nabla^2\,\Psi \;+\; 6\,\frac{a'}{a^3}\Big[\Psi'+\frac{a'}{a}\,\Phi\Big] \;+\; \bar{G}^0{}_0{}'(B-E')\ \ , \tag{119}$$

$$\Big(\Delta G^0{}_i\Big)_S \;=\; -\,\frac{2}{a^2}\,\partial_i\Big[\psi'+\frac{a'}{a}\,\phi\Big] \tag{120}$$

$$\;=\; -\,\frac{2}{a^2}\,\partial_i\Big[\Psi'+\frac{a'}{a}\,\Phi\Big] \;+\; \Big(\bar{G}^0{}_0-\frac{1}{3}\,\bar{G}^j{}_j\Big)\,\partial_i(B-E')\ \ , \tag{121}$$

$$\begin{aligned}\Big(\Delta G^i{}_j\Big)_S \;=\;& \delta_{ij}\times\frac{2}{a^2}\Big\{\Big[2\,\frac{a''}{a}-\frac{a'^{\,2}}{a^2}\Big]\phi \;+\; \frac{a'}{a}\Big[2\psi'+\phi'\Big] \;+\; \psi'' \\ &\qquad +\frac{1}{2}\nabla^2\Big[\phi-\psi-(B'-E'')-2\,\frac{a'}{a}(B-E')\Big]\Big\} \\ &+\frac{1}{a^2}\,\partial_i\partial_j\Big\{\psi-\phi+B'-E''+2\,\frac{a'}{a}(B-E'')\Big\}\end{aligned} \tag{122}$$

$$\begin{aligned}=\;& \frac{2}{a^2}\Big\{\frac{1}{2}\nabla^2(\Phi-\Psi)+\Psi''+\frac{a'}{a}\,(2\Psi'+\Phi')+\Big[2\,\frac{a''}{a}-\frac{a'^{\,2}}{a^2}\Big]\Phi\Big\} \\ &\times\delta_{ij} \;+\; \frac{1}{a^2}\,(\Psi_{,ij}-\Phi_{,ij}) \;+\; \bar{G}^i{}_j{}'(B-E')\ \ .\end{aligned} \tag{123}$$

(ii) The right hand side

A similar analysis must be made to derive $(\Delta T^{\mu}_{\nu})_S$. We need only compute the scalar perturbations contribution to Δp since the conservation equations will give the similar contribution to $\Delta\rho$ and Δu_i. [5] Instead of calculating directly from (78), we shall perform the computation sequentially by first considering the scalar perturbations contribution to (71):

$$\begin{aligned}\Big(\Delta R\Big)_S \;=\;& -12\,\frac{a''}{a^3}\,\phi \;-\; 6\,\frac{a'}{a^3}\Big[\phi'+3\psi'-\nabla^2(B-E')\Big] \\ &+\frac{2}{a^2}\Big[\nabla^2(2\psi-\phi)-3\psi''+\nabla^2(B'-E'')\Big]\end{aligned} \tag{124}$$

$$\begin{aligned}=\;& \frac{2}{a^2}\,\nabla^2(2\Psi-\Phi) \;-\; \frac{6}{a^2}\Big[\Psi''+\frac{a'}{a}\,(3\Psi'+\Phi')\Big] \\ &-2\bar{R}\,\phi \;+\; \frac{1}{a^2}\,(a^2\,\bar{R})'\,(B-E') \;+\; 2\bar{R}\,(B'-E'')\ \ ,\end{aligned} \tag{125}$$

[5] The perturbation Δu_0 does not enter $(\Delta T^{\mu}_{\nu})_S$ and, at any rate, has been calculated: $\Delta u_0=-a\phi$.

and the similar part of (76):

$$\Big[(\Delta\Box)\,\bar{X}\Big]_S \;=\; -\,2\bar{R}\,\phi \;+\; \frac{1}{a^2}\,\bar{X}'\,\Big[\,3\psi' + \phi' - \nabla^2(B-E')\,\Big] \tag{126}$$

$$\begin{aligned} &= \; -\,2\bar{R}\,\phi \;+\; \frac{1}{a^2}\,\bar{X}'\,\Big\{\,3\Psi' + \Phi' - \Big[\,2\,\frac{a''}{a} - 2\,\frac{a'^2}{a^2}\,\Big](B-E') \\ &\qquad -\frac{2a'}{a}\,(B'-E'') + (B''-E''') - \nabla^2(B-E')\,\Big\}\;. \end{aligned} \tag{127}$$

By exploiting (73), the last term in (127) can be rewritten as follows:

$$\begin{aligned} \frac{1}{a^2}\,\bar{X}'\,\nabla^2\Big[\,(B-E')\Big] \;&=\; \frac{1}{a^2}\,\nabla^2\,\Big[\,\bar{X}'\,(B-E')\,\Big] \\ &=\; \bar{\Box}\,\Big[\,\bar{X}'\,(B-E')\,\Big] \;+\; \frac{1}{a^2}\Big(\,\partial_0 + 2\frac{a'}{a}\,\Big)\partial_0\,\Big[\,\bar{X}'\,(B-E')\,\Big] \qquad (128) \\ &=\; \bar{\Box}\,\Big[\,\bar{X}'\,(B-E')\,\Big] \;-\; \frac{1}{a^2}\,(a^2\,\bar{R})'\,(B-E') \;-\; 2\bar{R}\,(B'-E'') \\ &\quad +\bar{X}'\,\Big\{\,\frac{1}{a^2}\,(B''-E''') \;-\; 2\,\frac{a'}{a^3}\,(B'-E'') \\ &\qquad -\Big[\,2\,\frac{a''}{a^3} - 2\,\frac{a'^2}{a^4}\,\Big](B-E')\,\Big\}\;, \qquad (129) \end{aligned}$$

where in the second step we used another direct consequence of (73):

$$-\;\frac{1}{a^2}\Big(\,\partial_0 + 2\frac{a'}{a}\,\Big)\partial_0\,\bar{X} \;=\; \bar{R} \qquad \Longrightarrow \qquad \bar{X}'' \;=\; -2\,\frac{a'}{a}\,\bar{X}' \;-\; a^2\,\bar{R}\;\;, \tag{130}$$

Appropriately combining equations (125, 127, 129), we get the scalar perturbations part of the source deviation:

$$\begin{aligned} \Big(\Delta X\Big)_S \;&=\; \frac{1}{\bar{\Box}}\,\Big[\,\Delta R - (\Delta\Box)\,\bar{X}\,\Big]_S \\ &=\; \bar{X}'\,(B-E') \;+\; \frac{1}{\bar{\Box}}\,\Big\{\,\frac{2}{a^2}\,\nabla^2(2\Psi-\Phi) \;-\; \frac{6}{a^2}\,\Big[\,\Psi'' + \frac{a'}{a}\,(3\Psi'+\Phi')\,\Big] \\ &\qquad -\frac{1}{a^2}\,\bar{X}'\,(3\Psi'+\Phi')\,\Big\}\;. \end{aligned} \tag{131}$$

The induced pressure first order scalar deviation follows from (67):

$$\begin{aligned} \Delta p \;=\; \bar{p}'\,(B-E') \;-\; \epsilon\Lambda^2\,f'[-\epsilon\,\bar{X}] \times \frac{1}{\bar{\Box}}\,\Big\{\,\frac{2}{a^2}\,\nabla^2(2\Psi-\Phi) \\ -\frac{6}{a^2}\,\Big[\,\Psi'' + \frac{a'}{a}\,(3\Psi'+\Phi')\,\Big] \;-\; \frac{1}{a^2}\,\bar{X}'\,(3\Psi'+\Phi')\,\Big\}\;. \end{aligned} \tag{132}$$

(iii) The conservation equations

The scalar perturbations part of the conservation equations (96, 98, 99) becomes respectively: [6]

$$\Big[a^3\,\Delta(\rho+p)\Big]' \;=\; a^3\left\{\Delta p' + (\bar\rho+\bar p)\Big[3\psi' - \nabla^2\Big(\frac{1}{a}\,\Delta u + B - E'\Big)\Big]\right\} \quad (133)$$

$$\left\{\Big[a^3(\bar\rho+\bar p)\Big]^2\,\Delta u^0\right\}' \;=\; -\left\{\Big[a^3(\bar\rho+\bar p)\Big]^2\,\frac{1}{a}\,\phi\right\}' \;\;, \quad (134)$$

$$\Big[a^3(\bar\rho+\bar p)\,\Delta u\Big]' \;=\; \Big[a^4(\bar\rho+\bar p)\,B\Big]' - a^4\Big[\Delta p + (\bar\rho+\bar p)\,\phi\Big] \;\;. \quad (135)$$

As before, equations (133, 135) determine Δu and $\Delta\rho$ while equation (134) is automaticaly satisfied by $\Delta u_0 = -a\phi$.

(iv) An observation

Consider the scalar perturbations part of the (ij) first order equation of motion:

$$\Big(\Delta G^i{}_j\Big)_S \;=\; 8\pi G\,\Big(\Delta T^i{}_j\Big)_S \;=\; 8\pi G\,(\Delta p)\,\delta^i{}_j \;\;. \quad (136)$$

When $i \neq j$, equation (136) simplifies considerably, since all terms proportional to δ_{ij} vanish, and allows us to conclude that $\Phi = \Psi$:

$$(i\neq j)_S \quad\Longrightarrow\quad \frac{1}{a^2}\,(\Psi_{,ij} - \Phi_{,ij}) = 0 \quad\Longrightarrow\quad \Phi = \Psi \;\;, \quad (137)$$

where we have used equations (123, 132) and the conformal time derivative of the (ij) background equation of motion:

$$\bar G^i{}_j{}' \;=\; 8\pi G\,\bar T^i{}_j{}' \;=\; 8\pi G\,\bar p'\,\delta^i{}_j \;\;. \quad (138)$$

$\star$ *Identities*

In addition to the equations presented throughout the main text, various of the following expressions have been used to obtain the results of this subsection:

$$h \;=\; -h_{00} + h_{ii} \;=\; 2\,\phi - 6\,\psi - 2\,\nabla^2 E \;\;, \quad (139)$$

$$h_{ii} \;=\; -6\,\psi - 2\,\nabla^2 E \;\;, \quad (140)$$

$$h_{0i,i} \;=\; -\nabla^2 B \;\;, \quad (141)$$

$$h_{ij,j} \;=\; -2\,\psi_{,i} - 2\,\nabla^2 E_{,i} \;\;, \quad (142)$$

$$h_{ij,ij} \;=\; -2\,\nabla^2\psi - 2\,\nabla^4 E \;\;, \quad (143)$$

[6] We have decomposed the perturbation Δu_i into its transverse and longitudinal parts: $\Delta u_i \equiv \Delta u_i^T + \partial_i\,\Delta u$. The transverse perturbation is of no concern to us.

$$\Phi' \; = \; \phi' \; - \; \left[\frac{a''}{a} - \frac{a'^{\,2}}{a^2} \right] (B - E') \; - \; \frac{a'}{a} \, (B' - E'') \; - \; (B'' - E''') \;\; , \qquad (144)$$

$$\Psi' \; = \; \psi' \; + \; \left[\frac{a''}{a} - \frac{a'^{\,2}}{a^2} \right] (B - E') \; + \; \frac{a'}{a} \, (B' - E'') \;\; , \qquad (145)$$

$$\Psi'' \; = \; \psi'' \; + \; \left[\frac{a'''}{a} - 3 \, \frac{a' a''}{a^2} + 2 \, \frac{a'^{\,3}}{a^3} \right] (B - E')$$

$$+ \left[2 \, \frac{a''}{a} - 2 \, \frac{a'^{\,2}}{a^2} \right] (B' - E'') \; + \; \frac{a'}{a} \, (B'' - E''') \;\; , \qquad (146)$$

$$\Psi' + \frac{a'}{a} \, \Phi \; = \; \psi' \; + \; \frac{a'}{a} \, \phi \; + \; \left[\frac{a''}{a} - \frac{a'^{\,2}}{a^2} \right] (B - E') \;\; , \qquad (147)$$

$$\Phi - \Psi \; = \; \phi - \psi - (B' - E'') - 2 \, \frac{a'}{a} \, (B - E') \;\; , \qquad (148)$$

$$3\Psi' + \Phi' \; = \; 3\psi' + \phi' + \left[2 \, \frac{a''}{a} - 2 \, \frac{a'^{\,2}}{a^2} \right] (B - E') + \frac{2a'}{a} \, (B' - E'')$$

$$- \, (B'' - E''') \;\; , \qquad (149)$$

$$\Psi'' + \frac{a'}{a} \, (3\Psi' + \Phi') \; = \; \psi'' \; + \; \frac{a'}{a} \, (3\psi' + \phi') \; + \; \frac{1}{6} \, (a^2 \, \bar{R})' \, (B - E')$$

$$+ \, \frac{a^2}{3} \, \bar{R} \, (B' - E'') \;\; , \qquad (150)$$

$$\bar{G}^0{}_0{}' \; = - \, 6 \, \frac{a'}{a^3} \left[\frac{a''}{a} - 2 \, \frac{a'^{\,2}}{a^2} \right] \;\; , \qquad (151)$$

$$\bar{G}^0{}_0 - \frac{1}{3} \, \bar{G}^j{}_j \; = \; \frac{2}{a^2} \left[\frac{a''}{a} - 2 \, \frac{a'^{\,2}}{a^2} \right] \;\; , \qquad (152)$$

$$\bar{G}^i{}_j{}' \; = \; - \, \frac{2}{a^2} \left[\frac{a'''}{a} - 4 \, \frac{a' \, a''}{a^2} + 2 \, \frac{a'^{\,3}}{a^3} \right] \delta^i{}_j \;\; , \qquad (153)$$

$$\bar{R}' \; = \; 6 \, \frac{a'''}{a^3} \; - \; 18 \, \frac{a' \, a''}{a^4} \qquad , \qquad (a^2 \, \bar{R})' \; = \; 6 \, \frac{a'''}{a} \; - \; 6 \, \frac{a' \, a''}{a^2} \;\; , \qquad (154)$$

$$\bar{p}' \; = \; -\epsilon \Lambda^2 \, \bar{X}' f'[-\epsilon \, \bar{X}] \;\; , \qquad (155)$$

$$\Delta u^i \; = \; \frac{1}{a} \, B_{,i} \; + \; \frac{1}{a^2} \, \Delta u_i \;\; . \qquad (156)$$

- **The Invariant Completions**

There is an elementary way to redefine the Einstein tensor perturbation

$\Delta G^\mu{}_\nu$ and the stress-energy tensor perturbation ΔT^μ_ν so that both sides of the equations of motion are gauge invariant [16]:

$$\Delta G^\mu{}_\nu \;=\; 8\pi G\, \Delta T^\mu_\nu \qquad \longrightarrow \qquad \Delta \mathcal{G}^\mu{}_\nu \;=\; 8\pi G\, \Delta \mathcal{T}^\mu_\nu \;\; . \tag{157}$$

The idea is for the differences between $(\Delta G^\mu{}_\nu\, ,\, \Delta T^\mu_\nu)$ and $(\Delta \mathcal{G}^\mu{}_\nu\, ,\, \Delta \mathcal{T}^\mu_\nu)$ to simultaneously make the latter gauge invariant and obey the gravitational equations of motion. This can be accomplished in general with the redefinitions:

$$\Delta \mathcal{G}^0{}_0 \;=\; \Delta G^0{}_0 \;-\; \bar{G}^0{}_0{}'(B - E') \;\; , \tag{158}$$

$$\Delta \mathcal{G}^0{}_i \;=\; \Delta G^0{}_i \;-\; \Big(\bar{G}^0{}_0 - \frac{1}{3}\,\bar{G}^j{}_j\Big)(B - E')_{,i} \;\; , \tag{159}$$

$$\Delta \mathcal{G}^i{}_j \;=\; \Delta G^i{}_j \;-\; \bar{G}^i{}_j{}'(B - E') \;\; . \tag{160}$$

For instance, in the case of scalar perturbations these difference terms are precisely the only terms of $(\Delta G^\mu_\nu)_S$ that are not gauge invariant as can be seen from (119, 121, 123). The corresponding redefinitions of the stress-energy perturbation are obvious:

$$\Delta \mathcal{T}^0_0 \;=\; \Delta T^0{}_0 \;-\; \bar{T}^0{}_0{}'(B - E') \;\; , \tag{161}$$

$$\Delta \mathcal{T}^0_i \;=\; \Delta T^0{}_i \;-\; \Big(\bar{T}^0_0 - \frac{1}{3}\,\bar{T}^j{}_j\Big)(B - E')_{,i} \;\; , \tag{162}$$

$$\Delta \mathcal{T}^i_j \;=\; \Delta T^i{}_j \;-\; \bar{T}^i{}_j{}'(B - E') \;\; . \tag{163}$$

(i) The scalar perturbations equations of motion

Returning to the scalar perturbations that concern us, we have: [7]

$$(00)_S \;\; \Longrightarrow \;\; \Delta \mathcal{G}^0{}_0 \;=\; 8\pi G \Big\{ \Delta T^0_0 \;-\; \bar{T}^0_0{}'(B - E') \Big\} \tag{164}$$

$$\Longrightarrow \;\; -\frac{2}{a^2}\,\nabla^2\Phi \;+\; 6\,\frac{a'}{a^3}\,\Phi' \;+\; 6\,\frac{a'^{\,2}}{a^4}\,\Phi \;=\; 8\pi G\,(-\Delta\mathcal{E}) \;\; , \tag{165}$$

$$(0i)_S \;\; \Longrightarrow \;\; \Delta \mathcal{G}^0{}_i \;=\; 8\pi G \Big\{ \Delta T^0_i \;-\; \Big(\bar{T}^0_0 - \frac{1}{3}\,\bar{T}^j{}_j\Big)(B - E')_{,i} \Big\} \tag{166}$$

$$\Longrightarrow \;\; -\frac{2}{a^2}\,\Phi' \;-\; 2\,\frac{a'}{a^3}\,\Phi \;=\; 8\pi G\, a^{-1}(\Delta\mathcal{U}) \;\; , \tag{167}$$

$$(ij)_S \;\; \Longrightarrow \;\; \Delta \mathcal{G}^i{}_j \;=\; 8\pi G \Big\{ \Delta T^i{}_j \;-\; \bar{T}^i{}_j{}'(B - E') \Big\} \tag{168}$$

$$\Longrightarrow \;\; \frac{2}{a^2}\,\Phi'' \;+\; 6\,\frac{a'}{a^3}\,\Phi' + \Big[\, 4\,\frac{a''}{a^3} - 2\,\frac{a'^{\,2}}{a^4}\,\Big]\,\Phi \;=\; 8\pi G\,(\Delta\mathcal{P}) \;\; . \tag{169}$$

[7] Since we are only concerned with scalar perturbations, from now on we shall not indicate this explicitly.

In arriving at equations (165, 167, 169) – besides the equality $\Phi = \Psi$ – we used expressions (119, 121, 123) respectively, as well as the definitions: [8]

$$\Delta\mathcal{E} \;\equiv\; \Delta\rho \;-\; \bar{\rho}'(B-E') \;\;, \tag{170}$$

$$\Delta\mathcal{U} \;\equiv\; \Delta u \;+\; a\,(B-E') \;\;, \tag{171}$$

$$\Delta\mathcal{P} \;\equiv\; \Delta p \;-\; \bar{p}'\,(B-E') \;\;. \tag{172}$$

Our induced pressure *ansatz* provides $\Delta\mathcal{P}$:

$$\Delta\mathcal{P} \;=\; -\,\epsilon\Lambda^2\, f'[-\epsilon\,\bar{X}]\times\frac{1}{\Box}\left\{\frac{2}{a^2}\,\nabla^2\Phi - \frac{6}{a^2}\left[\Phi''+\frac{4a'}{a}\,\Phi'\right] - \frac{4}{a^2}\,\bar{X}'\,\Phi'\right\} \tag{173}$$

The remaining two quantities $\Delta\mathcal{E}$, $\Delta\mathcal{U}$ are obtained from the conservation equations (175, 176).

(ii) The initial value problem

The appropriate set of initial value data for the system of equations of motion (165, 167, 169) consists of the following:

$$(\Delta\mathcal{E})_{\eta=\eta_0} \;\;\&\;\; (\Delta\mathcal{U})_{\eta=\eta_0} : \text{unrestricted} \qquad , \qquad (\Delta\mathcal{P})_{\eta=\eta_0} = 0 \;\;. \tag{174}$$

The requirement $(\Delta\mathcal{P})_{\eta_0} = 0$ comes from the perfect fluid form (2) of the stress-energy tensor and the induced pressure *ansatz* (3) ; $\Box^{-1}$ is the retarded Green's function and vanishes on the initial value surface. The initial value data $(\Delta\mathcal{E})_{\eta_0}$, $(\Delta\mathcal{U})_{\eta_0}$ are free and via equations (165, 167) determine Φ_{η_0}, Φ'_{η_0}. Then, equation (169) determines Φ''_{η_0}.

(iii) The scalar perturbations conservation equations

The equations of motion are augmented by the conservation equations (133, 135) whose invariant completions are:

$$\left[a^3\,\Delta\mathcal{E}\right]' \;=\; -\,(a^3)'\,\Delta\mathcal{P} \;+\; 3a^3\,(\bar{\rho}+\bar{p})\,\Phi' \;-\; a^3\,(\bar{\rho}+\bar{p})\,\nabla^2\Big(\frac{\Delta\mathcal{U}}{a}\Big) \;\;, \tag{175}$$

$$\left[a^3\,(\bar{\rho}+\bar{p})\,\Delta\mathcal{U}\right]' \;=\; -\,a^4\,\Delta\mathcal{P} \;-\; a^4\,(\bar{\rho}+\bar{p})\,\Phi \;\;. \tag{176}$$

(iv) The dynamical content

The fact that $(\Delta\mathcal{E})_{\eta_0}$ & $(\Delta\mathcal{U})_{\eta_0}$ are unrestricted and, therefore, Φ_{η_0} & Φ'_{η_0} are

[8]We have again decomposed the perturbation $\Delta\mathcal{U}_i$ into its transverse and longitudinal parts: $\Delta\mathcal{U}_i \equiv \Delta\mathcal{U}^T_i + \partial_i\,\Delta\mathcal{U}$.

also unrestricted, implies that there is a scalar degree of freedom which becomes dynamical due to the presence of our gravitationally induced stress-energy tensor $T^{\mu}_{\ \nu}[g]$.

This is to be contrasted with the situation where the origin of the stress-energy tensor is the matter sector of the theory; if we denote the relevant deviation of the latter tensor by $\Delta\Theta^{\mu}_{\ \nu}$, we have:

$$(00)_S \implies \quad -\frac{2}{a^2}\,\nabla^2\Phi \,+\, 6\,\frac{a'}{a^3}\Big[\,\Phi' + \frac{a'}{a}\,\Phi\,\Big] \;=\; 8\pi G\,\Delta\Theta^0_{\ \,0} \;\;, \tag{177}$$

$$(0i)_S \implies \quad -\frac{2}{a^2}\,\partial_i\Big[\,\Phi' + \frac{a'}{a}\,\Phi\,\Big] \;=\; 8\pi G\,\Delta\Theta^0_{\ \,i} \;=\; 8\pi G\,\partial_i(\Delta\Theta) \;\;, \tag{178}$$

where the last equality in (178) is true because we consider scalar perturbations. From (178) we immediately deduce:

$$\Phi' + \frac{a'}{a}\,\Phi \;=\; -\,\frac{a^2}{2}\,8\pi G\,\Delta\Theta \;\;. \tag{179}$$

Substituting (179) in the $(00)_S$ equation (177):

$$-\,\frac{2}{a^2}\,\nabla^2\Phi \,-\, 3\,\frac{a'}{a}\,8\pi G\,\Delta\Theta \;=\; 8\pi G\,\Delta\Theta^0_{\ \,0} \;\;, \tag{180}$$

allows us to conclude that Φ has no dynamics since we can solve for it from (180).

$\star$ *Identities*

In addition to the equations presented throughout the main text, various of the following expressions have been used to obtain the results of this subsection:

$$\bar{T}^0_{\ \,0}{}' \;=\; -\bar{\rho}' \;=\; -3\,\frac{a'}{a}\,(\bar{\rho}+\bar{p}) \;\;, \tag{181}$$

$$\bar{T}^i_{\ \,j}{}' \;=\; \bar{p}'\,\delta^i_{\ \,j} \qquad , \qquad \bar{T}^0_{\ \,0} - \frac{1}{3}\,\bar{T}^j_{\ \,j} \;=\; (\bar{\rho}+\bar{p}) \;\;, \tag{182}$$

4 Scalar Perturbations Equation Solutions

In this section we will investigate solutions of the non-local evolution equation (169). It is most convenient, for this purpose, to return to co-moving

coordinates and we shall do so:

$$\ddot{\Phi} + 4H\,\dot{\Phi} + (3H^2 + 2\dot{H})\,\Phi \;=\; -\,\omega^2 \times \frac{f'[-\epsilon\,\bar{X}]}{f'_{\rm cr}} \times \Bigl(-\frac{1}{\bar{\Box}}\Bigr) \Bigl\{ \ddot{\Phi} + 5H\,\dot{\Phi} + \frac{2}{3}\dot{\bar{X}}\,\dot{\Phi} + \frac{k^2}{3a^2}\,\Phi \Bigr\} \;, \qquad (183)$$

where the critical point $\bar{X}_{cr}$ and oscillation frequency ω are [13]:

$$1 \,-\, 8\pi\epsilon\, f[-\epsilon\,\bar{X}] \;=\; 0 \qquad , \qquad \omega^2 \;=\; 24\pi\,\epsilon^2 \Lambda\, f'_{cr} \;\; . \qquad (184)$$

The scalar d'Alembertian acting on a general function equals:

$$-\,\bar{\Box} \;=\; \partial_t^2 \,+\, 3H\,\partial_t \,+\, \frac{k^2}{a^2} \;\; . \qquad (185)$$

It should be clear that equation (183) cannot be solved exactly and we need to develop a methodology that will allow us to extract the time evolution of the scalar perturbations.

- **Strategy**

We can realize the inverse d'Alembertian using the mode functions $u(t,k)$ and $u^*(t,k)$ of the massless, minimally coupled scalar which obey the equations:

$$\ddot{u}(t,k) \,+\, 3H(t)\,\dot{u}(t,k) \,+\, \frac{k^2}{a^2(t)}\,u(t,k) \;=\; 0 \;\; , \qquad (186)$$

$$u(t,k)\,\dot{u}^*(t,k) \,-\, \dot{u}(t,k)\,u^*(t,k) \;=\; i\,a^{-3}(t) \;\; . \qquad (187)$$

The solutions for general $a(t)$ are quite complicated [20] but we shall only require the asymptotic forms long before and long after first horizon crossing:

$$k \gg H(t)a(t) \;\;\Longrightarrow\;\; u(t,k) \;\approx\; \frac{1}{\sqrt{2k}\;a(t)}\;\exp\Bigl[+ik\int_t^{+\infty}\frac{dt'}{a(t')}\Bigr] \;\; , \qquad (188)$$

$$k \ll H(t)a(t) \;\;\Longrightarrow\;\; u(t,k) \;\approx\; \frac{H(t_k)}{\sqrt{2k^3}}\Bigl\{\Bigl[1 + O(k^2)\Bigr] + \frac{i\,k^3}{H^2(t_k)}\Bigl[\int_t^{+\infty}\frac{dt'}{a^3(t')} + O(k^2)\Bigr]\Bigr\} \;\; , \qquad (189)$$

where t_k is the time of first horizon crossing:

$$k = H(t_k)\;a(t_k) \;\; . \qquad (190)$$

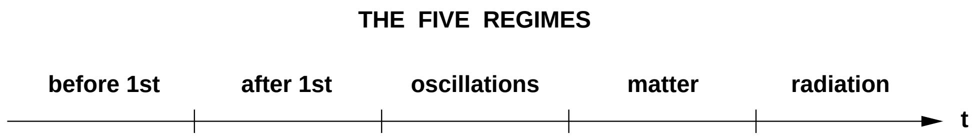

Figure 1: The regimes during which the evolution of scalar perturbations was studied.

We can construct the retarded Green's function of $\bar{\Box}$ using the mode functions $u(t,k)$ and $u^*(t,k)$. Hence the action of $-\Box^{-1}$ on some function $f(t,k)$ gives:

$$\Bigl(-\frac{1}{\bar{\Box}}\Bigr)\Bigl\{f\Bigr\}(t,k) \;\equiv\; i\int_0^t dt'\; a^3(t')\Bigl[\,u(t,k)\; u^*(t',k) - u^*(t,k)\; u(t',k)\,\Bigr]f(t',k) \;\;, \tag{191}$$

and further action on $-\bar{\Box}f(t,k)$ only gives back the function $f(t,k)$ up to homogeneous solutions:

$$\Bigl(-\frac{1}{\bar{\Box}}\Bigr)\Bigl\{-\bar{\Box}f\Bigr\}(t,k) \;=\; f(t,k) \,+\, \alpha\, u(t,k) \,+\, \beta\, u^*(t,k) \;\;. \tag{192}$$

The constants α and β can be expressed in terms of the initial values of f and its first derivatives:

$$\alpha \;=\; i\Bigl[\,\dot{u}^*(0,k)\; f(0,k) \;-\; u^*(0,k)\; \dot{f}(0,k)\,\Bigr] \;\;, \tag{193}$$

$$\beta \;=\; i\Bigl[\,u(0,k)\; \dot{f}(0,k) \;-\; \dot{u}(0,k)\; f(0,k)\,\Bigr] \;\;. \tag{194}$$

Our approach will be to divide time evolution into successive regimes and obtain reliable approximate solutions to (183) within each regime. The choice of these regimes is dictated by the actual physical evolution of the system and by our desire to reliably approximate (183). For instance, the last term can be irrelevant or important depending on whether the particular mode with wave number k has or has not experienced first horizon crossing respectively. We shall assume that the wave number k lies in the range for which the mode experiences first horizon crossing during inflation, but close enough to the end of inflation that the mode is at a cosmologically observable scale today. There are five epochs during which we seek approximate solutions to (183).

The strategy is always the same. We first determine the "f' ratio" in the left hand side of (183), then we approximate $\dot{\bar{X}}(t)$ in terms of $H(t)$. We next

identify the "large" part of the curly bracketed term in the right hand side of (183) and extract it from the inverse d'Alembertian using (192), along with the appropriate homogeneous solution.

- **The Inflationary Regime Before First Horizon Crossing**

The simplest epoch is the first one during which:

$$H^2(t) \ll \frac{k^2}{a^2(t)} \;\; , \;\; \omega^2 \ll H^2(t) \;\; , \;\; \dot{\bar{X}}(t) \approx -4H(t) \;\; , \;\; \frac{f'[-\epsilon \bar{X}(t)]}{f'[-\epsilon X_{cr}]} < 1 \quad (195)$$

and where the "f' ratio" is very much less than one for most choices of the function $f(x)$. The "large" part of the curly bracketed term is $\frac{k^2}{3a^2}\Phi$, so we extract it using (192, 185):

$$\Big(-\frac{1}{\bar{\square}}\Big) \Big\{ \ddot{\Phi} + 5H\,\dot{\Phi} + \frac{2}{3}\dot{\bar{X}}\,\dot{\Phi} + \frac{k^2}{3a^2}\,\Phi \Big\}(t,k) \approx$$
$$\frac{1}{3}\Phi(t,k) + \Big(\mathit{homogeneous}\Big) + \Big(-\frac{1}{\bar{\square}}\Big) \Big\{ \frac{2}{3}\ddot{\Phi} + \frac{4}{3}H\,\dot{\Phi} \Big\}(t,k) \quad (196)$$

where *(homogeneous)* stands for a linear combination of the homogeneous solutions $u(t,k)$ and $u^*(t,k)$ given by (188). Because of conditions (195) and the form of (188), the entire right-hand side of (183) is negligible and the equation reduces to:

$$\ddot{\Phi} + 4H\,\dot{\Phi} + (3H^2 + 2\dot{H})\,\Phi \approx 0 \;. \quad (197)$$

Approximate solutions of (197) are:

$$\Phi_1(t,k) \approx \frac{1}{a(t)} \qquad , \qquad \Phi_{1'}(t,k) \approx \frac{1}{a^3(t)} \;, \quad (198)$$

and the recognition of this fact allows us also to estimate the first corrections from the non-local term. Suppose, for instance, we are correcting the Φ_1 solution, in which case the non-local term is approximately given by the retarded Green's function (191) – constructed out of (188) – acting on the dominant term $\frac{k^2}{3a^2}\Phi_1$:

$$\Big(-\frac{1}{\bar{\square}}\Big) \Big\{ \ddot{\Phi} + 5H\,\dot{\Phi} + \frac{2}{3}\dot{\bar{X}}\,\dot{\Phi} + \frac{k^2}{3a^2}\,\Phi \Big\}(t,k)$$

$$\approx \frac{1}{k\,a(t)}\int_0^t dt'\,a^2(t')\,\sin\Bigl[k\int_{t'}^t \frac{dt''}{a(t'')}\Bigr] \times \frac{k^2}{3a^2(t')}\,\frac{1}{a(t')} \tag{199}$$

$$= \frac{1}{3a(t)}\int_0^t dt'\,\frac{k}{a(t')}\,\sin\Bigl[k\int_{t'}^t \frac{dt''}{a(t'')}\Bigr] \tag{200}$$

$$= \frac{1}{3a(t)}\Bigl\{1\,-\,\cos\Bigl[k\int_0^t \frac{dt'}{a(t')}\Bigr]\Bigr\}\quad . \tag{201}$$

The leading correction to Φ_1 derives from the non-oscillating term $\frac{1}{3a}$ in (201). However, this correction – which grows like $\omega^2 H^{-2}\ln a \times \Phi_1$ – is of the same kind as the correction to Φ_1 we would get if we take into account the $2\dot{H}\Phi$ term in equation (197). [9] Moreover, the dominant correction from the oscillating term in (201) is, again, computed by inserting it as the source in the evolution equation (183) and noting that the leading time dependence comes from the oscillating source term itself:

$$\Delta\Phi_1(t,k) \;=\; \Phi_1(t,k)\;\times\;\frac{f'[-\epsilon\,\bar{X}(t)]}{f'[-\epsilon\,X_{cr}]}\;\times\Bigl(-\frac{1}{3}\Bigr)\Bigl(\frac{\omega\,a(t)}{k}\Bigr)^2\cos\Bigl[k\int_0^t \frac{dt'}{a(t')}\Bigr] \tag{202}$$

The correction is very small long before first horizon crossing, owing to the factor of $k^{-2}\omega^2 a^2(t)$. However, we see that the oscillatory term does grow.

- **The Inflationary Regime After First Horizon Crossing**

During this regime we are close enough to the end of inflation that, for instance, the "f' ratio" is nearly unity:

$$\frac{k^2}{a^2(t)} \ll H^2(t) \;\;,\;\; \omega^2 \ll H^2(t) \;\;,\;\; \dot{\bar{X}}(t) \approx -4H(t) \;\;,\;\; \frac{f'[-\epsilon\bar{X}(t)]}{f'[-\epsilon X_{cr}]} \approx 1 \tag{203}$$

This is a difficult epoch to understand from first principles so we had recourse to explicit numerical studies. These revealed an end to the fall off in $\Phi(t,k)$ which characterizes the previous epoch. In fact, the solution changes sign and its magnitude seems then to grow slowly. One can understand why this happens from the constant homogeneous solution which is built up by integrating the retarded Green's function over times t' long before first horizon crossing. For this case we approximate $u(t,k)$ by expression (189) and $u(t',k)$ by expression (188). We further assume $\Phi(t',k) \approx a^{-1}(t')$, and that

[9] During inflation $\dot{H} \approx -\frac{2}{9}\omega^2$.

the $\frac{k^2}{3a^2}\Phi(t')$ term dominates: [10]

$$\Big(-\frac{1}{\bar{\Box}}\Big)\Big\{\ddot{\Phi} + 5H\,\dot{\Phi} + \frac{2}{3}\dot{\bar{X}}\,\dot{\Phi} + \frac{k^2}{3a^2}\,\Phi\Big\}(t,k)$$

$$\approx \frac{H(t_k)}{k^2}\int_0^{t_k} dt'\,a^2(t')\,\sin\Big[k\int_{t'}^{+\infty}\frac{dt''}{a(t'')}\Big]\times\frac{k^2}{3a^2(t')}\,\frac{1}{a(t')} \tag{204}$$

$$= \frac{1}{3a(t_k)}\Big\{\cos\Big[k\int_{t_k}^{+\infty}\frac{dt'}{a(t')}\Big] - \cos\Big[k\int_0^{+\infty}\frac{dt'}{a(t')}\Big]\Big\} \equiv C_2(k) \;\;, \tag{205}$$

and we see that the constant is positive: $C_2(k) > 0$. Thus the approximate evolution equation during this epoch is:

$$\ddot{\Phi} + 4H\,\dot{\Phi} + (3H^2 + 2\dot{H})\,\Phi \approx -\omega^2\,C_2(k) \;\; . \tag{206}$$

Even though $\omega^2 \ll H^2(t)$, the term on the right is not zero, so the $a^{-1}(t)$ fall off of $\Phi(t,k)$ cannot persist indefinitely. When this rapid time evolution of $\Phi(t,k)$ comes to an end the time derivative terms become insignificant and, because $-\dot{H}(t) \ll H^2(t)$, we have:

$$3H^2(t)\,\Phi(t,k) \approx -\omega^2\,C_2(k) \quad\Longrightarrow\quad \Phi(t,k) \approx -\frac{\omega^2\,C_2(k)}{3H^2(t)} \;\; . \tag{207}$$

Hence the solution changes sign and, because $H(t)$ decreases slowly during inflation, the magnitude of the solution increases slowly. That is exactly what the numerical simulations show.

- **The Oscillatory Regime**

During the epoch of oscillations the "f' ratio" is still unity, and we also have:

$$\frac{k^2}{a^2(t)} \ll H^2(t) \ll |\dot{H}| \ll \omega^2 \;\;,\;\; \dot{\bar{X}}(t) \approx -6H(t) \;\;,\;\; \frac{f'[-\epsilon\bar{X}(t)]}{f'[-\epsilon X_{cr}]} \approx 1 \tag{208}$$

Expression (207) implies that $\Phi(t,k)$ must begin evolving again at the end of inflation, so that its time derivatives are no long negligible. This has two consequences: first, the non-local term receives substantial contributions

[10]The contribution to the integration in (204) from t_k to t is subleading relative to that from 0 to t_k.

from times t' after criticality; and second, the "large" term is $\ddot{\Phi}$. Therefore we can write:

$$\Big(-\frac{1}{\bar{\square}}\Big)\Big\{\,\ddot{\Phi} \,+\, 5H\,\dot{\Phi} \,+\, \frac{2}{3}\dot{\bar{X}}\,\dot{\Phi} \,+\, \frac{k^2}{3a^2}\,\Phi\,\Big\}(t,k) \;\approx\; \Phi(t,k) \,+\, C_3(k) \quad , \quad (209)$$

where $C_3(k)$ consists of $C_2(k)$ plus a new, time independent contribution. Employing (208), the mode equation becomes effectively:

$$\ddot{\Phi}(t,k) \,+\, 4H(t)\,\dot{\Phi}(t,k) \,+\, \omega^2\,\Phi(t,k) \approx -\omega^2\,C_3(k) \qquad (210)$$

$$\Longrightarrow \quad \Phi(t,k) \;\approx\; -C_3(k) \,+\, \frac{\Phi_3(k)\sin[\,\omega\Delta t + \phi_3(k)\,]}{a^2(t)} \quad , \quad (211)$$

where $\Delta t \,\equiv\, t - t_{\rm cr}$ and $\Phi_3(k)$, $\phi_3(k)$ are constants. Had we included the first correction to the right hand side of (209) the result would have been to slightly modify the rate of fall off in the oscillatory term, but it would not change the frequency. We emphasize that *all* the super-horizon modes oscillate at this same frequency. This is a profound distinction between the scalar perturbations of our model and those of scalar-driven inflation, and it has important consequences for reheating.

- **The Subsequent Matter and Radiation Domination Regimes**

Since the field Φ couples to matter universally with gravitational strength, its oscillations with frequency ω will most likely excite the particles with masses $m \sim \omega$. These will be very heavy particles which will behave – even after their excitation – like non-relativistic matter. The very heavy particles that were the primary receptors of the energy from the oscillating field Φ will quickly decay into a hot radiation dominated universe.

The analysis for these two epochs of matter is much the same. The only difference concerns the approximation we make for $\dot{\bar{X}}(t)$:

$$\textit{matter} \quad \Longrightarrow \quad \dot{\bar{X}}(t) \;\approx\; -2H(t) \quad , \qquad (212)$$

$$\textit{radiation} \quad \Longrightarrow \quad \dot{\bar{X}}(t) \;\approx\; 0 \quad . \qquad (213)$$

These differences shows up only in the correction to (209), which affects the rate of fall off but not the oscillatory frequency. Furthermore, more evolution also affects the constant parts accumulated from the homogeneous solution:

$$\textit{matter} \quad \Longrightarrow \quad \Phi(t,k) \;\approx\; -C_4(k) \,+\, \frac{\Phi_4(k)\,\sin[\,\omega\Delta t + \phi_4(k)\,]}{a^2(t)} \quad , \quad (214)$$

$$\textit{radiation} \quad \Longrightarrow \quad \Phi(t,k) \;\approx\; -C_5(k) \,+\, \frac{\Phi_5(k)\,\sin[\,\omega\Delta t + \phi_5(k)\,]}{a^2(t)} \quad . \quad (215)$$

We have *not* included the very significant decline in amplitude which must occur due to the flow of energy from the scalar modes into normal matter. It seems clear that this will continue until the amplitude of oscillation is driven to nearly zero. The final signal for the power spectrum resides in the constant $C_5(k)$ whose normalization we cannot fix in the absence of canonical quantization.

5 The Normalization of Perturbations

Our effective field equations govern the time dependence of perturbations. We cannot actually solve these equations exactly but let us suppose, for the purposes of this discussion, that we could. For each wave vector $\mathbf{k}$ that would determine two linearly independent solutions, $\Phi_1(t,\mathbf{k})$ and $\Phi_2(t,\mathbf{k})$. The full content of the effective field equations has been exhausted by expressing the perturbation operator $\widetilde{\Phi}(t,\mathbf{k})$ as a linear combination of these two solutions:

$$\widetilde{\Phi}(t,\mathbf{k}) \;=\; \alpha_1(\mathbf{k}) \times \Phi_1(t,\mathbf{k}) \;+\; \alpha_2(\mathbf{k}) \times \Phi_2(t,\mathbf{k}) \;\; . \tag{216}$$

We can say what the operator coefficients, $\alpha_1(\mathbf{k})$ and $\alpha_2(\mathbf{k})$, are in terms of the initial values of $\widetilde{\Phi}(t,\mathbf{k})$ and its first derivative, but the field equations alone do not define how these operators commute. That information would ordinarily derive from applying canonical quantization to a Lagrangian, but in our non-local cosmological model we have no Lagrangian. We must instead regard the missing information as a separate assumption, which can be specified however we wish. It is a fundamental part of the definition of the model, every bit as much as the effective field equations were.

Before stating this assumption, let us clarify the issues in the very simple context of a 1-dimensional point particle whose position $q(t)$ obeys the simple harmonic oscillator equation:

$$\ddot{q}(t) + \omega^2 q(t) \;=\; 0 \;\; . \tag{217}$$

This is a trivial equation to solve, and we can use it to express $q(t)$ in terms of its initial values q_0 and $\dot{q}_0$:

$$q(t) \;=\; q_0 \cos(\omega t) + \frac{\dot{q}_0}{\omega} \sin(\omega t) \;\; . \tag{218}$$

By decomposing the oscillatory functions into positive and negative frequencies we can identify linear combinations of the initial value operators which

must lower and raise the energy:

$$q(t) \;=\; \frac{1}{2}\Big(q_0 + \frac{i\dot{q}_0}{\omega}\Big)\, e^{-i\omega t} \;+\; \frac{1}{2}\Big(q_0 - \frac{i\dot{q}_0}{\omega}\Big)\, e^{i\omega t}$$

$$\Rightarrow \quad \Big[\, H \,,\, q_0 \pm \frac{i\dot{q}_0}{\omega} \,\Big] \;=\; \mp\, \hbar\omega\Big(q_0 \pm \frac{i\dot{q}_0}{\omega}\Big) \;. \tag{219}$$

Relation (219) is as far as one can go using only the equation of motion (217). We do not know how q_0 and $\dot{q}_0$ commute, nor do we know how the Hamiltonian depends on them. Indeed, these two issues are intimately related. If we ignore possible operator ordering ambiguities, the two Hamiltonian evolution equations:

$$\dot{q}_0 \;=\; \frac{i}{\hbar}\,[H, q_0] \;=\; -\frac{i}{\hbar}\,\frac{\partial H}{\partial \dot{q}_0}\,[q_0, \dot{q}_0] \;\;, \tag{220}$$

$$-\omega^2 q_0 \;=\; \frac{i}{\hbar}\,[H, \dot{q}_0] \;=\; \frac{i}{\hbar}\,\frac{\partial H}{\partial q_0}\,[q_0, \dot{q}_0] \;\;, \tag{221}$$

are consistent with any solution of the form:

$$H = F(\mathcal{E}) \quad \text{and} \quad [q_0, \dot{q}_0] \;=\; \frac{i\hbar}{F'(\mathcal{E})} \quad , \text{ where } \;\; \mathcal{E} \;\equiv\; \frac{1}{2}\dot{q}_0^2 \,+\, \frac{1}{2}\omega^2 q_0^2 \;\; . \tag{222}$$

The equation of motion (217) cannot tell us what the function $F(\mathcal{E})$ is.

Note that there is still an ambiguity even if we assume $F(\mathcal{E})$ is linear – which assumption might seem reasonable (athough not necessary) in view of the fact that the equation of motion is linear. The ambiguity rests with the proportionality constant: any function of the form $F(\mathcal{E}) = K\mathcal{E}$ would reproduce the canonical operator equations (220-221). [11] Therefore, if we write $q(t)$ as a linear combination of canonically normalized creation and annihilation operators, the amplitude with which they appear contains a factor of the arbitrary constant K:

$$q(t) \;=\; \sqrt{\frac{\hbar}{2K\omega}}\,\Big\{a\, e^{-i\omega t} \,+\, a^\dagger\, e^{i\omega t}\Big\} \quad , \quad [a, a^\dagger] \;=\; 1 \;\; . \tag{223}$$

It is canonical quantization of the simple harmonic oscillator Lagrangian which would ordinarily fix this constant:

$$L \;=\; \frac{1}{2} m\, \dot{q}^2 \,-\, \frac{1}{2} m\omega^2 q^2 \qquad \Longrightarrow \qquad K = m \;\; . \tag{224}$$

[11] There is no operator ordering ambiguity for any linear ansatz.

In our case, the effective field equations are not associated with any Lagrangian. Indeed, the simultaneously causal and non-local nature of our equations precludes their derivation from any single-field Lagrangian. [12] A correct derivation from fundamental theory would involve an effetive action of the Schwinger-Keldysh type in which more than one quantity stands for what will eventually be the same dynamical variable, and one obtains the equation of motion by varying first and only then setting the different quantities equal [21].

In the absence of an action principle we are forced to regard specification of the constant K as an independent assumption, with the same status as the effective field equations themselves. This assumption can be made however we wish as part of how we define the model. In this context it should be noted that no principle seems to preclude the constant depending upon the magnitude of the wave vector, $k = \|\mathbf{k}\|$. [13] It is immediately obvious that we can enforce both the observed magnitude of scalar perturbations, and their approximate scale invariance, if only the super-horizon mode functions freeze in to constants in the late time regime. We saw in the previous section that the super-horizon mode functions do become constant during radiation domination, so that is the non-trivial check of our class of models, not either the magnitude nor the approximate scale invariance of scalar perturbations.

These comments need not apply to tensor perturbations. The equations which describe the latter are the same as those of general relativity, provided the expansion history $a(t)$ is fixed. We can therefore invoke the usual canonical normalization of the graviton creation and annihilation operators even though that is not, strictly speaking, required. Doing so makes the full tensor power spectrum a prediction of our model [17].

6 Reheating

We have seen from Section 4 that the evolution of scalar perturbations reveals a profound difference between our model and any scalar-driven model of inflation:

[12] Although a partial integration "trick" can give causal non-local field equations, the resulting models inevitably suffer an undesirable renormalization of the effective Newton's constant which the present class of models avoids by construction [2].

[13] Invariance under spatial rotations rules out more general dependence.

- In scalar-driven inflation the scalar mode functions oscillate and decay until horizon crossing, after which they approach constants; whereas
- The scalar mode functions of our model decay until horizon crossing, then they become approximately constant until the end of inflation, after which they oscillate – all with about the same frequency ω – until enough energy has been dumped into the matter sector to support a radiation dominated cosmology.

Another key distinction is that, whereas no one knows or cares how long scalar-driven inflation persists beyond the 50-60 e-foldings needed to solve the horizon and flatness problems, our model requires the vast number of $(G\Lambda)^{-1} \gtrsim 10^6$ e-foldings of inflation. This implies that the number density n of super-horizon modes at the end of inflation is staggering:

$$n \sim \frac{H^3}{3\pi^2} \exp\Bigl(\frac{3}{G\Lambda}\Bigr) \gtrsim H^3 \times 10^{10^6} \; . \tag{225}$$

Now consider what must happen when all of these modes start oscillating. There is not much energy in any one mode, and they couple only gravitationally to ordinary matter, so energy flow from them is very weak. However, the number density (225) of modes which participate is so enormous that reheating must be practically instantaneous.

It is interesting to contrast reheating in our model with the way it works in traditional single-scalar inflation. In those models the single inflaton zero mode undergoes oscillations at the end of inflation. At this stage it becomes necessary to assume that the inflaton couples to ordinary matter because having the its kinetic energy transferred to gravity and, then, from gravity to ordinary matter, would be far too slow to reheat the universe. What would happen instead is that the energy would be reshifted away before it had a chance to accumulate and thermalize.

Direct couplings between the inflaton and ordinary matter can result in significant reheating. However, it comes at the price of more fine tuning because such couplings cause loops of matter quanta to induce Coleman-Weinberg terms in the inflaton effective potential $V_{\rm eff}$. These contributions would not be in terms of naturally small parameters, they would necessarily involve the relatively large coupling constants of ordinary matter. For example, a cubic coupling between the inflaton φ and either a spectator scalar χ

or a fermion field ψ would induce [18]:

$$- \, g_\chi \, \varphi \chi^2 \quad \Longrightarrow \qquad \Delta V_{\rm eff} \; = \; + \frac{g_\chi^2 \, \varphi^4}{16\pi^2} \, \ln\Bigl(\frac{\varphi^2}{\mu^2}\Bigr) \;\; , \tag{226}$$

$$- \, g_\psi \, \varphi \, \overline{\psi} \psi \quad \Longrightarrow \qquad \Delta V_{\rm eff} \; = \; - \frac{g_\psi^2 \, \varphi^4}{16\pi^2} \, \ln\Bigl(\frac{\varphi^2}{\mu^2}\Bigr) \;\; , \tag{227}$$

for some renormalization scale μ. Either of these contributions would render the inflaton potential far too steep for successful inflation; the fermionic contribution (227) would actually make the inflaton unstable. Hence these effects must be canceled by adding carefully chosen terms to the classical action. As long as one is not restricted to renormalizable inflaton potentials this can be done, but it represents a new level of fine tuning.

This additional challenge for scalar-driven inflation derives from the fact that reheating draws its energy from the oscillations of just a single scalar zero mode. Thus, the inflaton must be directly coupled to ordinary matter to give efficient reheating. By contrast, the reheating in our model draws its energy from the vast reservoir of super-horizon modes which are naturally accumulated during the long epoch of inflation. Because so many modes participate, it is not necessary (or even possible) to introduce a new, direct coupling to ordinary matter; gravitational couplings will suffice.

7 Epilogue

The phenomenological model considered in this study is solely based on the graviton and therefore, one would argue, it should have major problems in reproducing *any* realistic scalar density perturbations; after all, the graviton is a tensor field possessing four unconstrained initial value data which result in its two physical polarizations. Unless we are willing to invoke graviton bound states, there is simply no physical scalar degree of freedom present. However, this argument ignores the presence of the gravitationally induced non-local source term in the field equations. As we showed in Section 3, its presence changes the dynamical content of the theory and, besides the two graviton polarizations, the scalar Φ emerges as a physical degree of freedom possessing two unconstrained initial value data: $\Delta\mathcal{E}_0$ and $\Delta\mathcal{U}_0$.

Deriving the evolution equation for the scalar Φ is a non-trivial exercise described in detail in Section 3. The (approximate) solutions to the equation in five successive regimes of evolution from the onset of inflaton until late

SCALAR MODE FUNCTIONS BEHAVIOUR		
EPOCH	**STANDARD**	**MODEL**
from: **onset of inflation** ***until:*** **1st crossing**	**falling**	**falling**
from: **1st crossing** ***until:*** **end of inflation**	**constant**	**constant**
oscillations	——	**oscillating**
matter	**constant**	**oscillating**
radiation	**constant**	**constant**

Figure 2: The mode functions evolution in scalar-driven *vs* gravity-driven cosmology.

times, showed distinctive differences with the standard inflationary picture; they are qualitatively recorded in Figure 2. What counts is agreement with measurements and what is measured is correlations between different portions of the sky. These correlations are non-zero for the scalar part of the generic perturbations in the gravitational theory. Scalar-driven inflation makes a non-zero contribution dictated by the form of the inflaton Lagrangian. Our gravity-driven model also makes a non-zero contribution which – with the proper normalization choice – is consistent with the observed magnitude and approximate scale invariance of the scalar spectrum.

The novel feature of the gravity-driven model is the presence of an era subsequent to inflation during which *all* modes of Φ oscillate. By all modes we mean all infrared modes since Φ is a dynamical degree of freedom emerging in the infrared sector of the theory. Now the "receiver" of the energy generated by the oscillations will be the matter sector of the theory. If radiation domination is reached – and our model is predisposed to do so [19] because radiation is the unique power law solution for which our simple source vanishes ($R = 0$) – the energy deposited into matter will sustain the radiation

domination.

Each mode will contribute a very small amount of energy to the process but since there exist a huge amount of modes the whole process can be very efficient. Moreover, this reheating mechanism is very natural because the interaction of the modes with any matter field is of the *universal* gravitational strength. In contradistinction, scalar-driven theories involve different couplings of the inflaton to different matter fields. In one sentence, the scalar Φ via its coherent oscillations can naturally reheat and lead to a hot thermal universe.

The simple phenomenological model used in this paper has late time evolution problems which can be addressed by modifying the *ansatz* for the gravitationally induced source [19]. This change, however, only affects the late time evolution and does not disturb the results of this paper concerning primordial scalar perturbations.

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